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Efficient Growth with Exhaustible Resources in a Neoclassical Model

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1. INTRODUCTION

Since Malinvaud's paper [6], various aspects of intertemporal efficiency have been the subject of extensive discussion. The important special models, that have received the most attention, however, do not emphasize the role of exhaustible resources as factors of production. (For a recent exception, see Stiglitz [9].) "Natural resources" are often assumed to be supplied exogenously in given quantities in each period. Clearly, such an approach can study the role of primary factors like labor, but is unsuitable for capturing the essence of problems involved in the best use of a nonaugmentable, storable, resource, whose availability in a particular period depends strictly on how much has already been used up in the past.

I shall consider here a model of intertemporal allocation, which pays particular attention to the significance of an exahustible resource as a factor of production, and study the properties of efficient growth programs in this framework. In order to expose the basic features, I shall keep the model simple. I shall assume that there is one produced good, which can either be used for consumption or for further production. A primary factor (labor), appears explicitly in the production function. So does an exhaustible resource, the total supply of which is given. The use of the resource over the (infinite) planning horizon must not exceed this available supply. It should be noted that this framework is a particular case of the general models of intertemporal allocation, presented by, for instance, Malinvaud [6], Radner [7], and Cass [2], if we treat the exhaustible resource as a special type of capital good. (For a detailed explanation, see Section 2.)

By focusing on the special properties associated with exhaustible resources, one can obtain results which are sharper than those proved in the general models. The main results are summarized below:

(1) An interesting duality implication of the analysis is that along a competitive program (and, therefore, along an efficient program) the present value price of the exhaustible resource is a constant (Theorem 3.1).

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(2) Under the assumption that the exhaustible resource is an "important" factor of production (see Sect. 4, (A3)), the following characterization of efficiency is obtained. An interior program is efficient if and only if

(a) it is competitive,

(b) it satisfies the transversality condition (i.e., the value of the resource stock plus the value of capital stock converges to zero) (Theorem 4.1).

All the assumptions under which this conclusion holds are satisfied, in particular, by a Cobb–Douglas production function, which, in turn, makes the model a particular case of the one considered by Cass [1], provided the exhaustible resource is left out. Thus, clearly, the introduction of the resource leads to the following qualitative difference in the characterization of efficiency. Inefficiency is signalled by the transverslaity condition (b) being violated, and not by the condition of Cass, viz., that "the terms of trade from the present to the future deteriorate at a sufficiently rapid rate as the future recedes into the distance."

(3) Finally, we note that, if the primary factor (labor) is bounded away from zero, and bounded above, then the presence of the exhaustible resource implies that there is no competitive overaccumulation of capital, i.e., the augmentable good (Corollary 6.1). This result, together with an example (see Example 5.1), establishes that only overaccumulation of the exhaustible resource (and not overaccumulation of the augmentable good) is consistent with competitive pricing.

2. The Model

Consider an economy with a technology given by a production function, G, from R_+^3 to R_+. The production possibilities consist of capital input, K, exhaustible resource input, R, labor input, L, and current output $Z = G(K, R, L)$, for $(K, R, L) \geqslant 0$.[1]

Capital is considered to depreciate at a constant rate, δ, where $0 \leqslant \delta \leqslant 1$. The production function, G, and the depreciation rate, δ, together define a total output $Y = G(K, R, L) + (1 - \delta) K$ for $(K, R, L) \geqslant 0$. A *total output function*, F, can then be defined by

$$F(K, R, L) = G(K, R, L) + (1 - \delta) K \quad \text{for} \quad (K, R, L) \geqslant 0. \tag{2.1}$$

The production function, G, is assumed to satisfy:

(A1) *$G(K, R, L)$ is concave, homogeneous of degree one, and twice continuously differentiable, for $(K, R, L) \gg 0$.*

[1] For any two n-vectors, u and v, $u \geqslant v$ means $u_i \geqslant v_i$ for $i = 1,..., n$; $u > v$ means $u \geqslant v$ but $u \neq v$; $u \gg v$ means $u_i > v_i$ for $i = 1,..., n$.

(A2) $(G_K, G_R, G_L) \gg 0$ *for* $(K, R, L) \gg 0$.[2]

The initial capital input, **K**, and the initial available stock of the exhaustible resource, **S**, are considered to be historically given, and positive. The available labor force is assumed to be positive, and exogenously given at each date and denoted by $\mathbf{L}_t$ for $t \geqslant 0$.

A *feasible program* is a sequence $\langle K, R, L, Y, C\rangle = \langle K_t, R_t, L_t, Y_{t+1}, C_{t+1}\rangle$ satisfying

$$\begin{aligned} &K_0 = \mathbf{K}, \quad \sum_{t=0}^{\infty} R_t \leqslant \mathbf{S}, \quad L_t = \mathbf{L}_t && \text{for} \quad t \geqslant 0, \\ &Y_{t+1} = F(K_t, R_t, L_t), \quad C_{t+1} = Y_{t+1} - K_{t+1} && \text{for} \quad t \geqslant 0, \\ &(K_t, R_t, L_t, Y_{t+1}, C_{t+1}) \geqslant 0 && \text{for} \quad t \geqslant 0. \end{aligned} \tag{2.2}$$

Associated with a feasible program $\langle K, R, L, Y, C\rangle$ is a *sequence of resource stocks* $S = \langle S_t\rangle$, given by

$$S_0 = \mathbf{S}, \qquad S_t = \mathbf{S} - \sum_{\tau=0}^{t-1} R_\tau \qquad \text{for} \quad t \geqslant 1. \tag{2.3}$$

By (2.2), $S_t \geqslant 0$, for $t \geqslant 0$. A feasible program $\langle K, R, L, Y, C\rangle$ is *interior* if $(K_t, R_t) \gg 0$ for $t \geqslant 0$.

A feasible program $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C}\rangle$ is called *short-run inefficient* if there is a feasible program $\langle K, R, L, Y, C\rangle$, and $T < \infty$, such that

$$(C_1, ..., C_{T+1}, K_{T+1}, S_{T+1}) > (\bar{C}_1, ..., \bar{C}_{T+1}, \bar{K}_{T+1}, \bar{S}_{T+1}). \tag{2.4}$$

A feasible program is *short-run efficient* if it is not short-run inefficient. Thus, a short-run efficient program yields maximal consumption, terminal capital stock, and terminal resource stock, for *every* finite horizon.

A feasible program $\langle K, R, L, Y, C\rangle$ *dominates* a feasible program $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C}\rangle$ if $C_t \geqslant \bar{C}_t$ for all $t \geqslant 1$, and $C_t > \bar{C}_t$ for some t. A feasible program $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C}\rangle$ is *inefficient* if there is a feasible program $\langle K, R, L, Y, C\rangle$ which dominates it. It is *efficient* if it is not inefficient. It is, of course, immediate, in our framework, that an efficient program is also short-run efficient. The converse, however, is not true. For a detailed discussion of these concepts, see Cass [2].

It will be noticed that what has been called an "exhaustible resource" above, is really a capital good, with the properties that (i) net investment output of that good is always nonpositive, and (ii) output of the capital-cum-consumption good depends just on the net investment output, and not on the stock, of that good.[3] Under this interpretation, it is useful to view

[2] G_K denotes $\partial G(K, R, L)/\partial K$; similarly, for G_R, G_L.

[3] I shall, however, continue to refer to K as the "capital good," and R as the "resource," to distinguish the two, in discussions to follow. One could call K "augmentable capital" and R "exhaustible capital," or "nonaugmentable capital." This latter terminology, although more precise, is somewhat cumbersome, and shall not be used.

the production possibilities, in the "stock version," as given by a technology set $\mathscr{T}$ of input–output paris in the following way:

$$\mathscr{T} = \{[(K, S, L), (Y, S', 0)]: O \leqslant Y \leqslant F(K, R, L); \\ R \leqslant (S - S'); (K, R, L, S') \geqslant 0\}. \tag{2.5}$$

It is, of course, clear that for a feasible program $\langle K, R, L, Y, C \rangle$, $[(K_t, S_t, L_t), (Y_{t+1}, S_{t+1}, 0)] \in \mathscr{T}$ for $t \geqslant 0$.

A feasible program $\langle K, R, L, Y, C \rangle$ is called *competitive*, if there is a non-null sequence of nonnegative vectors $\langle p, q, w \rangle = \langle p_t, q_t, w_t \rangle$ such that, for $t \geqslant 0$,

$$p_{t+1}Y_{t+1} + q_{t+1}S_{t+1} - p_tK_t - q_tS_t - w_tL_t \\ \geqslant p_{t+1}Y + q_{t+1}S' - p_tK - q_tS - w_tL \\ \text{for all } [(K, S, L), (Y, S', 0)] \in \mathscr{T} \ldots \tag{2.6}$$

In other words, the intertemporal profit maximization condition (2.6) is satisfied at each date.

We can associate with a competitive program $\langle K, R, L, Y, C \rangle$, a *capital value sequence* $U = \langle U_t \rangle$ given by

$$U_t = p_tK_t \qquad \text{for} \quad t \geqslant 0, \tag{2.7}$$

a *resource value sequence* $V = \langle V_t \rangle$ given by

$$V_t = q_tS_t \qquad \text{for} \quad t \geqslant 0, \tag{2.8}$$

and a *wages sequence* $W = \langle W_t \rangle$ given by

$$W_t = w_tL_t \qquad \text{for} \quad t \geqslant 0. \tag{2.9}$$

A competitive program $\langle K, R, L, Y, C \rangle$ is said to satisfy the *transversality condition* if

$$\lim_{t \to \infty} (U_t + V_t) = 0. \tag{2.10}$$

It is said to satisfy the *capital value transversality condition* if

$$\lim_{t \to \infty} U_t = 0. \tag{2.11}$$

Finally, it is said to have *finite consumption value* if

$$\sum_{t=1}^{\infty} p_tC_t < \infty. \tag{2.12}$$

3. Competitive and Short-Run Efficient Programs

In this section, I shall show that along interior competitive programs, the present-value price of the exhaustible resource is constant. This result is really not surprising, when the resource is viewed as a special type of capital good. In view of the fact that the output of the capital-cum-consumption good depends just on the net investment output $(-R_t)$, and not on the stock (S_t) of the resource, the net present rental for the stock of the resource (call it r_t) is zero in the asset-making-clearing equation: $q_t = q_{t+1} + r_t$, provided that the resource has not been completely exhausted, in finite time.

With the help of this result, I shall demonstrate the equivalence (for interior programs) of the competitive condition and the condition of short-run efficiency.

The results of this section can be viewed in the following way. Even if one is interested only in short-run efficiency, there is a straightforward implication for the appropriate pricing of the exhaustible resource, viz. its current price should be increasing at the rate of interest, as determined by the net marginal product of capital $(G_{K_t} - \delta)$.

Theorem 3.1. *Under* (A1), (A2), *if an interior program* $\langle K, R, L, Y, C\rangle$ *is competitive, then* $q_t = q_{t+1}$ *for* $t \geqslant 0$.

Proof. If $\langle K, R, L, Y, C\rangle$ is competitive, then there is $\langle p, q, w\rangle$ such that (2.6) holds for $t \geqslant 0$. Since G is homogeneous of degree one, so $\mathscr{T}$ is a cone, and the left-hand side of (2.6) is zero for $t \geqslant 0$.

Since $[(0, S, 0), (0, S, 0)] \in \mathscr{T}$, for $S > 0$, we have $q_{t+1}S - q_tS \leqslant 0$, implying $q_{t+1} \leqslant q_t$ for $t \geqslant 0$.

Now, suppose $q_{t+1} < q_t$ for some t. Since $\langle K, R, L, Y, C\rangle$ is interior, so $S_t > R_t > 0$. Choose $\bar{S}$, such that $S_t > \bar{S} > R_t$. Then, surely, $[(K_t, \bar{S}, L_t); (Y_{t+1}, \bar{S} - R_t, 0)] \in \mathscr{T}$, and using this in (2.6), we get

$$q_{t+1}(S_t - R_t) - q_tS_t \geqslant q_{t+1}(\bar{S} - R_t) - q_t\bar{S}. \tag{3.1}$$

(3.1) implies that $(q_{t+1} - q_t)(S_t - \bar{S}) \geqslant 0$, which contradicts $q_{t+1} < q_t$. Hence, $q_t = q_{t+1}$ for $t \geqslant 0$.

It is useful to look at the competitive condition (2.6) in a somewhat different way. In the following proposition, I shall show the equivalence between the competitive condition, and a condition which says that the proportional *rate of change* of the marginal productivity of the resource always equals the *level* of the (net) marginal productivity of capital.[4] This

[4] Condition (3.2) states that $(F_{R_{t+1}}/F_{R_t}) = F_{K_{t+1}}$, so that $[(G_{R_{t+1}} - G_{R_t})/G_{R_t}] + 1 = G_{K_{t+1}} + (1 - \delta)$. This means $(G_{R_{t+1}} - G_{R_t})/G_{R_t} = (G_{K_{t+1}} - \delta)$. The expression $(G_{K_{t+1}} - \delta)$ is called net marginal productivity of capital, in the discussion.

latter condition, in effect, says that the return from leaving a dollar's worth of the resource in the ground (a capital gain) equals the return from renting a dollar's worth of capital. Thus, along a competitive program, the capital good and the resource, are equally attractive earning assets (at the margin).

PROPOSITION 3.1. *Under* (A1), (A2), *an interior program* $\langle K, R, L, Y, C\rangle$ *is competitive if and only if*

$$F_{R_{t+1}} = F_{R_t} \cdot F_{K_{t+1}} \quad \text{for} \quad t \geqslant 0. \tag{3.2}$$

Proof. (*Sufficiency*). By concavity of F, we have, for $(K, R, L) \geqslant 0$, $F(K, R, L) - F(K_t, R_t, L_t) \leqslant F_{K_t}(K - K_t) + F_{R_t}(R - R_t) + F_{L_t}(L - L_t)$. Using this, we have for $t \geqslant 0$,

$$\begin{aligned} &F(K_t, R_t, L_t) - F_{K_t} \cdot K_t - F_{R_t} \cdot R_t - F_{L_t} \cdot L_t \\ &\quad \geqslant F(K, R, L) - F_{K_t} \cdot K - F_{R_t} \cdot R - F_{L_t} \cdot L \quad \text{for} \quad (K, R, L) \geqslant 0\ldots. \end{aligned} \tag{3.3}$$

Define

$$\begin{aligned} p_0 &= (F_{K_0}/F_{R_0}); \qquad p_{t+1} = (p_t/F_{K_t}) \quad \text{for} \quad t \geqslant 0; \\ w_t &= F_{L_t} \cdot p_{t+1} \quad \text{for} \quad t \geqslant 0; \qquad q_t = 1 \quad \text{for} \quad t \geqslant 0. \end{aligned} \tag{3.4}$$

Now, multiplying (3.3) by p_{t+1}, and using (3.2), we have, for $t \geqslant 0$,

$$\begin{aligned} &p_{t+1}F(K_t, R_t, L_t) - p_tK_t - q_tR_t - w_tL_t \\ &\quad \geqslant p_{t+1}F(K, R, L) - p_tK - q_tR - w_tL \quad \text{for} \quad (K, R, L) \geqslant 0. \end{aligned} \tag{3.5}$$

Rewriting this in a different way, we obtain, for $t \geqslant 0$,

$$\begin{aligned} &p_{t+1}Y_{t+1} + q_{t+1}S_{t+1} - p_tK_t - q_tS_t - w_tL_t \\ &\quad \geqslant p_{t+1}Y + q_{t+1}S' - p_tK - q_tS - w_tL \quad \text{for} \quad [(K, S, L), (Y, S', 0)] \in \mathscr{T}. \end{aligned} \tag{3.6}$$

(*Necessity*). If $\langle K, R, L, Y, C\rangle$ is interior and competitive, then $q_t = q_{t+1} = \hat{q}$ (say) for $t \geqslant 0$ (by Theorem 3.1). Hence, for $t \geqslant 0$,

$$\begin{aligned} &p_{t+1}F(K_t, R_t, L_t) - p_tK_t - \hat{q}R_t - w_tL_t \\ &\quad \geqslant p_{t+1}F(K, R, L) - p_tK - \hat{q}R - w_tL \quad \text{for} \quad (K, R, L) \geqslant 0. \end{aligned} \tag{3.7}$$

Since intertemporal profit $[p_{t+1}F(K, R, , L) - p_tK - \hat{q}R - w_tL]$ is maximized at an interior point, so for $t \geqslant 0$,

$$p_{t+1}F_{K_t} = p_t; \qquad p_{t+1}F_{R_t} = \hat{q}; \qquad p_{t+1}F_{L_t} = w_t. \tag{3.8}$$

Since $\langle p, q, w\rangle$ is nonnull, by (A2), we have $p_t > 0$, $w_t > 0$ for $t \geqslant 0$, and $\hat{q} > 0$. Hence, by (3.8),

$$\frac{F_{R_{t+1}}}{F_{R_t}} = \frac{p_{t+1}F_{R_{t+1}}}{p_{t+1}F_{R_t}} = \frac{p_{t+1}}{p_{t+2}} \frac{p_{t+2}F_{R_{t+1}}}{p_{t+1}F_{R_t}} = \frac{p_{t+1}}{p_{t+2}} = F_{K_{t+1}}, \tag{3.9}$$

which establishes (3.2) for $t \geqslant 0$.

Remark 3.1. A sidelight of Proposition 3.1 is that, for interior competitive programs, using (A2), we can ensure the prices $(p_t, q_t, w_t) \gg 0$ for $t \geqslant 0$. Since G is differentiable, the price sequence $\langle p, q, w\rangle$ is determined uniquely (up to positive scalar multiplication) and can be expressed in terms of the marginal products of the various inputs, as in (3.4). In view of this, when I refer to an interior competitive program, in the rest of the paper, I shall mean that it satisfies the intertemporal profit maximization condition (2.5) at precisely the prices, defined in (3.4).

PROPOSITION 3.2. *Under* (A1), (A2), *an interior program* $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C}\rangle$ *is short-run efficient if and only if it is competitive.*

Proof. (*Necessity*) For each T $(T \geqslant 1)$, a short-run efficient program $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C}\rangle$ solves the following nonlinear programming problem: Maximize $(\mathrm{S} - \sum_{t=0}^{T} R_t)$, subject to

(i) $C_{t+1} \geqslant \bar{C}_{t+1}$ for $t = 0, 1,\ldots, T$,

(ii) $K_{T+1} \geqslant \bar{K}_{T+1}$,

(iii) condition (2.2).

It can be checked that the above can be converted into a Lagrangian saddle-point problem, unconstrained with respect to (i), (ii), noting that the constraint qualification of Karlin [4] is satisfied. Using multipliers μ_{t+1}, $t = 0,\ldots, T$ and ν_{T+1}, the Lagrangian is a function of these, and by (iii), of the choice variable K_{t+1}, R_t, $t = 0, 1,\ldots, T$. The Lagrangian is

$$H(\cdot) = \left(\mathrm{S} - \sum_{t=0}^{T} R_t\right) + \sum_{t=0}^{T} \mu_{t+1}(C_{t+1} - \bar{C}_{t+1}) + \nu_{T+1}(K_{T+1} - \bar{K}_{T+1}).$$

Since $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C}\rangle$ is interior, the solution values are strictly positive. Hence,

$$\partial L/\partial R_t = (-1) + \mu_{t+1}F_{\bar{R}_t} = 0 \qquad \text{for } 0 \leqslant t \leqslant T,$$

$$\partial L/\partial K_{t+1} = \mu_{t+1}(-1) + \mu_{t+2}F_{\bar{K}_{t+1}} = 0 \qquad \text{for } 0 \leqslant t \leqslant T-1,$$

$$\partial L/\partial K_{T+1} = \mu_{T+1}(-1) + \nu_{T+1} = 0.$$

Simplifying, we get (since the multipliers are positive from above), for $0 \leqslant t \leqslant T-1$

$$F_{\bar{R}_{t+1}} = F_{\bar{R}_t} \cdot F_{\bar{K}_{t+1}}. \tag{3.10}$$

We claim that (3.10) holds for $t \geqslant 0$. For, suppose it were violated for some $t = \tau$. Then choose the horizon T, in the above exercise, to be $(\tau + 1)$. We get an immediate contradiction. This establishes the claim, and, by Proposition 3.1, proves necessity.

(*Sufficiency*) If $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C} \rangle$ is competitive, then it satisfies (2.6) for $t \geqslant 0$, and $\langle \bar{p}, \bar{q}, \bar{w} \rangle$ are as defined in (3.4) [see Remark 3.1]. Suppose $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C} \rangle$ is not short-run efficient. Then, there is a feasible program $\langle K, R, L, Y, C \rangle$ satisfying (2.4) for some $T \geqslant 0$. Since $\bar{q}_t = 1$ for $t \geqslant 0$, and $\bar{p}_t > 0$ for $t \geqslant 0$, so

$$\sum_{t=0}^{T} \bar{p}_{t+1} C_{t+1} + \bar{p}_{T+1} K_{T+1} + \left(\mathbf{S} - \sum_{t=0}^{T} R_t\right)$$
$$> \sum_{t=0}^{T} \bar{p}_{t+1} \bar{C}_{t+1} + \bar{p}_{T+1} \bar{K}_{T+1} + \left(\mathbf{S} - \sum_{t=0}^{T} \bar{R}_t\right). \tag{3.11}$$

Using (2.6), we have, however,

$$\sum_{t=0}^{T} \bar{p}_{t+1}(C_{t+1} - \bar{C}_{t+1}) \leqslant \sum_{t=0}^{T} (R_t - \bar{R}_t) + \bar{p}_{T+1}(\bar{K}_{T+1} - K_{T+1}), \tag{3.12}$$

which contradicts (3.11). This proves sufficiency.

4. A Complete Characterization of Efficiency

In this section, I shall establish that an interior program is efficient if and only if (a) it is competitive, and (b) it satisfies the transversality condition.

First, we need a definition. The *elasticity of current output with respect to the exhaustible resource*, or, briefly, the *resource elasticity*, is defined as:

$$\eta_R = \frac{G_R \cdot R}{G(K, R, L)} \quad \text{for} \quad (K, R, L) \gg 0. \tag{4.1}$$

The resource is called *important* in production if there is $\alpha > 0$, such that

$$\eta_R \geqslant \alpha > 0 \quad \text{for} \quad (K, R, L) \gg 0. \tag{4.2}$$

I shall assume, in this section, that

(A3) *the resource is important in production.*

We note that (A1), (A2), (A3) are satisfied by a Cobb–Douglas production function.[5]

The procedure adopted to prove the necessity part of the characterization theorem, is to use (A3) to show that an efficient program has finite consumption value. The sufficiency part is, of course, well known in the literature, and needs no proof.

THEOREM 4.1. *Under* (A1), (A2), (A3), *an interior program* $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C}\rangle$ *is efficient if and only if*

(a) *it is competitive, and*

(b) *it satisfies the transversality condition.*

Proof. (*Sufficiency*) This follows[6] from Malinvaud [6], Lemma 5.

(*Necessity*) Suppose the interior program $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C}\rangle$ is efficient. Then, it is short-run efficient. Hence, by Proposition 3.2, it is competitive, at a price sequence $\langle p, q, w\rangle$, defined as in (3.4).

Note that $\langle \bar{S}_t\rangle$ is a nonincreasing sequence, bounded below, so it has a limit. Also, $\lim_{t\to\infty} \bar{S}_t = 0$ must be satisfied. For, if $\lim_{t\to\infty} \bar{S}_t > 0$, then $\sum_{t=0}^{\infty} \bar{R}_t < \mathbf{S}$, and we could use an extra $(\mathbf{S} - \sum_{t=0}^{\infty} \bar{R}_t)$ amount in the period $t = 0$, produce and consume more in period 1, leaving the rest of the program unaffected. This would violate efficiency.

Since $\bar{q}_t = 1$ for $t \geqslant 0$, we have only to prove that the capital value transversality condition is satisfied, in order to ensure that (b) is satisfied.

First, we prove that $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C}\rangle$ has finite consumption value and that $\langle \bar{p}_t\bar{K}_t\rangle$ is a convergent sequence. Using (a), we have, for $T \geqslant 0$,

$$\sum_{t=0}^{T} \bar{p}_{t+1}\bar{C}_{t+1} = \bar{p}_0\mathbf{K} + \sum_{t=0}^{T} \bar{R}_t + \sum_{t=0}^{T} \bar{w}_t\bar{L}_t - \bar{p}_{T+1}\bar{K}_{T+1}\,, \tag{4.3}$$

since G is homogeneous of degree one. Now, we evaluate $\sum_{t=0}^{T} \bar{w}_t\bar{L}_t$. By definition of $\bar{w}_t$ and (A3),

$$\bar{w}_t\bar{L}_t = \left[\frac{G_{\bar{L}_t}}{G_{\bar{R}_t}}\right]\bar{L}_t = \frac{(G_{\bar{L}_t}\cdot\bar{L}_t)/G(\bar{K}_t, \bar{R}_t, \bar{L}_t)}{(G_{\bar{R}_t}\cdot\bar{R}_t)/G(\bar{K}_t, \bar{R}_t, \bar{L}_t)\,\bar{R}_t} \leqslant \frac{\bar{R}_t}{\alpha}, \tag{4.4}$$

[5] More generally, if $G(K, R, L)$ can be written in separable form as $f(K)g(R)h(L)$, then (A3) is satisfied whenever $\inf_{\mathbf{S}\geqslant R>0} (g'(R)\cdot R/g(R)) > 0$, and $f(K) > 0$ for $K > 0$, $h(L) > 0$ for $L > 0$. Thus, in particular, $g(R)$ could be of the form $R/(1 + R)$, or more generally, $R/[R^{\beta} + 1]^{1/\beta}$, where $\beta > 0$.

[6] $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C}\rangle$ is interior and competitive, so it is intertemporal profit maximizing at $\langle \bar{p}, \bar{q}, \bar{w}\rangle$, defined as in (3.4), so $\bar{p}_t > 0$ for $t \geqslant 0$. (See Remark 3.1.)

so that $\sum_{t=0}^{T} \bar{w}_t\bar{L}_t \leqslant \sum_{t=0}^{T} \bar{R}_t/\alpha \leqslant \mathbf{S}/\alpha$. Hence, $\sum_{t=0}^{\infty} \bar{w}_t\bar{L}_t$ is convergent, and, so, by (4.3), $\sum_{t=0}^{\infty} \bar{p}_{t+1}\bar{C}_{t+1}$ is convergent.[7] Since all other terms in (4.3) are convergent, $\langle \bar{p}_t\bar{K}_t\rangle$ is a convergent sequence.

To prove (b), we have to show that $\lim_{t\to\infty} \bar{p}_t\bar{K}_t = 0$. Suppose, on the contrary, that this limit is not zero. Then, there exists a $\hat{\beta} > 0$, such that, for $t \geqslant 0$, $\bar{p}_t\bar{K}_t \geqslant \hat{\beta} > 0$. Since $\sum_{t=0}^{\infty} \bar{p}_{t+1}\bar{C}_{t+1}$ is convergent, so there is $T_1 < \infty$, such that

$$\bar{J}_{T_1} = \sum_{t=T_1}^{\infty} \bar{p}_{t+1}\bar{C}_{t+1} \leqslant \hat{\beta}/2. \tag{4.5}$$

If $\bar{J}_{T_1} = 0$, the given program is clearly inefficient. So, we consider only the case in which $\bar{J}_{T_1} \neq 0$. Let $x_{t+1} = (\bar{p}_{t+1}\bar{C}_{t+1})/\bar{J}_{T_1}$ for $t \geqslant T_1$. Clearly, $\sum_{t=T_1}^{\infty} x_{t+1} = 1$. Now, construct a sequence $\langle K, R, L, Y, C\rangle$ in the following way. Let $K_0 = \mathbf{K}$, $R_0 = \bar{R}_0$; $(K_t, R_t, Y_t, C_t) = (\bar{K}_t, \bar{R}_t, \bar{Y}_t, \bar{C}_t)$ for $1 \leqslant t < T_1$. For $t = T_1$, $K_t = \frac{1}{2}\bar{K}_t$, $R_t = \frac{1}{2}\bar{R}_t$, $C_t = \bar{C}_t + \frac{1}{2}\bar{K}_t$, $Y_t = \bar{Y}_t$. For $t > T_1$, $K_t = \bar{K}_t[\frac{1}{2} - \frac{1}{2}\sum_{s=T_1}^{t-1} x_{s+1}] = \lambda_t\bar{K}_t$, $R_t = \lambda_t\bar{R}_t$, $Y_t = F(K_{t-1}, R_{t-1}, L_{t-1})$, $C_t = Y_t - K_t$. Finally, $L_t = \bar{L}_t$ for $t \geqslant 0$. (Note that $0 \leqslant \lambda_t \leqslant \frac{1}{2}$.)

Now, clearly, $(K_t, R_t) \gg 0$ for $t \geqslant 0$; $C_t = \bar{C}_t$ for $t < T_1$, and $C_t > \bar{C}_t$ for $t = T_1$. We will show that $C_t \geqslant 0$ for $t > T_1$ (so that $\langle K, R, L, Y, C\rangle$ is a feasible program) and that $C_t \geqslant \bar{C}_t$ for $t > T_1$, so that $\langle K, R, L, Y, C\rangle$ dominates $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C}\rangle$. This will contradict the hypothesis that $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C}\rangle$ is efficient, and establish that $\lim_{t\to\infty} \bar{p}_t\bar{K}_t = 0$, and, hence necessity.

For $t > T_1$, we have

$$\begin{aligned}
\bar{p}_tC_t &= \bar{p}_tF(K_{t-1}, R_{t-1}, L_{t-1}) - \bar{p}_tK_t \\
&\geqslant \bar{p}_tF(\lambda_{t-1}\bar{K}_{t-1}, \lambda_{t-1}\bar{R}_{t-1}, \lambda_{t-1}\bar{L}_{t-1}) - \bar{p}_t\lambda_t\bar{K}_t \\
&= \lambda_{t-1}\,\bar{p}_tF(\bar{K}_{t-1}, \bar{R}_{t-1}, \bar{L}_{t-1}) - \lambda_t\,\bar{p}_t\bar{K}_t \\
&= \lambda_{t-1}\,\bar{p}_t(\bar{C}_t + \bar{K}_t) - \lambda_t\,\bar{p}_t\bar{K}_t \\
&= \lambda_{t-1}\,\bar{p}_t\bar{C}_t + (1 - \lambda_{t-1})\,\bar{p}_t\bar{C}_t + \lambda_{t-1}\,\bar{p}_t\bar{K}_t - \lambda_t\,\bar{p}_t\bar{K}_t - (1 - \lambda_{t-1})\,\bar{p}_t\bar{C}_t \\
&\geqslant \bar{p}_t\bar{C}_t + \bar{p}_t\bar{K}_t[\lambda_{t-1} - \lambda_t - \tfrac{1}{2}(1 - \lambda_{t-1})\,x_t] \\
&\geqslant \bar{p}_t\bar{C}_t + \bar{p}_t\bar{K}_t[\tfrac{1}{2}x_t - \tfrac{1}{2}x_t] \\
&= \bar{p}_t\bar{C}_t.
\end{aligned}$$

Since $\bar{p}_t > 0$ for $t \geqslant 0$, we have $C_t \geqslant \bar{C}_t \geqslant 0$ for $t > T_1$. This proves that $\langle K, R, L, Y, C\rangle$ is a feasible program, and that it dominates $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C}\rangle$,

[7] This is the crucial step in the proof. After having established this, alternative methods can be employed to complete the rest of the argument, than the one presented here. Majumdar suggested that an adaptation of the proof of Theorem 2.1 in Majumdar et al. [5] can be used. Cass pointed out that the corollary to the theorem in Cass and Yaari [3] can also be used, by suitably redefining the production function, at each point of time.

proving that the latter is inefficient. This contradiction establishes that $\lim_{t\to\infty} \bar{p}_t \bar{K}_t = 0$, and, hence, condition (b).

Remark 4.1. (i) David Cass pointed out to me that, in a simpler version of the model examined here, viz. where capital is absent as a factor of production, (A3) is *necessary and sufficient* for every efficient program to have finite consumption value. Since it is this particular property, which is crucial to the technique of proof of Theorem 4.1, his demonstration seems to provide ample "justification" for the assumption.

However, a similar statement *cannot* be made with the model examined here, where capital, in fact, is *not* absent. There seems to be some room for relaxing (A3), and yet characterizing efficiency in terms of the conditions (a) and (b) of Theorem 4.1, in view of the following example. Consider $G(K, R, L) = [K^{1/2} + R^{1/2} + L^{1/2}]^2$; $\delta = 0$, $\mathbf{L}_t = 1$ for $t \geqslant 0$. Clearly $\eta_R \to 0$ as $R \to 0$, when $L = 1$, so that (A3) is violated. However, it is easy to check that for an interior efficient program $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C} \rangle$, $\sum_{t=0}^{\infty} \bar{p}_{t+1} \bar{C}_{t+1} < \infty$.

(ii) It is clear that the transversality condition holds if and only if (b1) the capital-value transversality condition holds and (b2) $\lim_{t\to\infty} \bar{S}_t = 0$. In the following discussions, I shall refer to (b2) as the *resource exhaustion condition*, since it says that $\sum_{t=0}^{\infty} \bar{R}_t = \mathbf{S}$.

The following two corollaries are immediate from Theorem 4.1, and, so, they are stated without proofs. Corollary 4.1 may be interpreted in the following way. If we treat initial stocks, $\mathbf{S}$, $\mathbf{K}$, as primary factors, the transversality condition can be replaced by the condition that the present value of the consumption sequence equals the present value of all primary factors.

COROLLARY 4.1. *Under* (A1), (A2), (A3), *an interior program* $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C} \rangle$ *is efficient if and only if*

(a) *it is competitive, and*

(b) $\infty > \sum_{t=0}^{\infty} \bar{p}_{t+1} \bar{C}_{t+1} = \bar{p}_0 \mathbf{K} + \mathbf{S} + \sum_{t=0}^{\infty} \bar{w}_t \bar{L}_t$.

Corollary 4.2 states that efficiency is equivalent to maximizing the present value of the consumption sequence, at a sequence of positive prices, in the set of all feasible consumption sequences.

COROLLARY 4.2. *Under* (A1), (A2), (A3), *an interior program* $\langle \bar{K}, \bar{R}, \bar{L}, \bar{Y}, \bar{C} \rangle$ *is efficient if and only if there exists a price sequence* $\bar{p} = \langle \bar{p}_t \rangle$, $\bar{p}_t > 0$, *such that*

$$\infty > \sum_{t=0}^{\infty} \bar{p}_{t+1} \bar{C}_{t+1} \geqslant \sum_{t=0}^{\infty} \bar{p}_{t+1} C_{t+1}$$

for every feasible program $\langle K, R, L, Y, C \rangle$.

5. Two Examples of Competitive Inefficiency

It should be clear, from Section 4, that, in our framework, competitive inefficiency can arise in one of two ways, viz. by violation of either (i) the resource exhaustion condition, or (ii) the capital-value transversality condition. In this section, we shall provide an example of each of these types of competitive inefficiency.

Example 5.1. An interior competitive program which satisfies the capital-value transversality condition, but violates the resource exhaustion condition.

Let $G(K, R, L) = K^{1/2}R^{1/4}L^{1/4}$ and $\delta = 0$; $\mathbf{L}_t = 1$ for $t \geqslant 0$; $\mathbf{K} = 1$, $\mathbf{S} = 4$.

Consider the sequence $\langle K, R, L, Y, C\rangle$ given by $K_t = 1$ for $t \geqslant 0$, $L_t = \mathbf{L}_t$ for $t \geqslant 0$, $R_0 = 1$, $\frac{1}{2}R_{t+1} + R_{t+1}^{3/4} = R_t^{3/4}$ for $t \geqslant 0$, and $C_{t+1} = G(K_t, R_t, L_t)$ for $t \geqslant 0$.

To show that $\langle K, R, L, Y, C\rangle$ is a feasible program, we have only to check that $\sum_{t=0}^{\infty} R_t \leqslant \mathbf{S}$. Note that $\frac{1}{2}R_{t+1} = R_t^{3/4} - R_{t+1}^{3/4}$ for $t \geqslant 0$, so that $\frac{1}{2}\sum_{t=0}^{T} R_{t+1} = R_0^{3/4} - R_{T+1}^{3/4}$. Hence $\sum_{t=0}^{\infty} R_t$ is convergent, and $R_t \to 0$ as $t \to \infty$. So, $\frac{1}{2}\sum_{t=0}^{\infty} R_{t+1} = R_0^{3/4} = 1$, i.e.,

$$\sum_{t=0}^{\infty} R_{t+1} = 2, \qquad \text{and} \qquad \sum_{t=0}^{\infty} R_t = R_0 + \sum_{t=0}^{\infty} R_{t+1} = 1 + 2 = 3 \leqslant \mathbf{S}.$$

It can be checked, now, that $\langle K, R, L, Y, C\rangle$ satisfies (i) the competitive conditions, and (ii) the capital value transversality condition. To check (i), note that

$$F_{R_{t+1}} = \tfrac{1}{4}R_{t+1}^{-3/4}K_{t+1}^{1/2} = \tfrac{1}{4}R_{t+1}^{-3/4};$$

and

$$F_{R_t} \cdot F_{K_{t+1}} = (\tfrac{1}{4}R_t^{-3/4})(\tfrac{1}{2}R_{t+1}^{1/4}K_{t+1}^{-1/2} + 1) = \tfrac{1}{4}[\tfrac{1}{2}R_{t+1} + R_{t+1}^{3/4}]^{-1}[\tfrac{1}{2}R_{t+1}^{1/4} + 1]$$
$$= \tfrac{1}{4}R_{t+1}^{-3/4}[\tfrac{1}{2}R_{t+1}^{1/4} + 1]^{-1}[\tfrac{1}{2}R_{t+1}^{1/4} + 1] = \tfrac{1}{4}R_{t+1}^{-3/4}.$$

So, by Proposition 3.1, $\langle K, R, L, Y, C\rangle$ is competitive. To check (ii), note that $p_{t+1}K_{t+1} = 1/F_{R_t} = 4R_t^{3/4}$, so we have $\lim_{t\to\infty} p_{t+1}K_{t+1} = \lim_{t\to\infty} 4R_t^{3/4} = 0$.

Finally, observe that the resource exhaustion condition is violated, since $\sum_{t=0}^{\infty} R_t = 3 < 4 = \mathbf{S}$.

Example 5.2. *An interior competitive program which satisfies the resource exhaustion condition, but violates the capital-value transversality condition.*

Let $G(K, R, L) = K^{1/4}R^{1/2}L^{1/4}$, and $\delta = 0$; $\mathbf{L}_t = [1/2^t]^4$, $\mathbf{K} = 1$, $\mathbf{S} = M$, where M is defined in the following way. Consider the sequences $\langle \Theta_t \rangle$ and $\langle d_t \rangle$ defined simultaneously by: $d_0 = 1$; $\Theta_t = [2^{t+2} \prod_{s=0}^{t} (2 + d_s)]^{-1}$ for $t \geqslant 0$, and $d_{t+1} = -1 + (1 + \Theta_t)^{1/2}$ for $t \geqslant 0$. Clearly the two sequences are positive sequences. Now, the series $\sum_{t=0}^{\infty} [1/\prod_{s=0}^{t} (2 + d_s)]^2$ is clearly convergent. The sum of this series is called M.

Consider the sequence $\langle K, R, L, Y, C \rangle$ given by: $L_t = \mathbf{L}_t$ for $t \geqslant 0$; $K_t = 1$ for $t \geqslant 0$; $R_t = [1/\prod_{s=0}^{t} (2 + d_s)]^2$ for $t \geqslant 0$, and $C_{t+1} = G(K_t, R_t, L_t)$ for $t \geqslant 0$. Noting the manner in which $\mathbf{S}$ was defined, $\langle K, R, L, Y, C \rangle$ is a feasible program.

It can be checked now that $\langle K, R, L, Y, C \rangle$ satisfies (i) the competitive conditions and (ii) the resource exhaustion condition. To check (i), note that

$$F_{R_{t+1}} = \tfrac{1}{2} R_{t+1}^{-1/2} K_{t+1}^{1/4} L_{t+1}^{1/4} = \{\tfrac{1}{2} R_{t+1}^{-1/2}\}/2^{t+1} = \tfrac{1}{2} \left\{ \prod_{s=0}^{t+1} (2 + d_s) \right\} \Big/ 2^{t+1}.$$

Similarly,

$$F_{R_t} = \tfrac{1}{2} \left\{ \prod_{s=0}^{t} (2 + d_s) \right\} \Big/ 2^t.$$

Also,

$$F_{K_{t+1}} = [1 + \tfrac{1}{4} R_{t+1}^{1/2} K_{t+1}^{-3/4} L_{t+1}^{1/4}] = \left[1 + \tfrac{1}{4} \left\{ 2^{t+1} \prod_{s=0}^{t+1} (2 + d_s) \right\}^{-1} \right].$$

So

$$F_{R_t} \cdot F_{K_{t+1}} = \tfrac{1}{2} \left\{ \prod_{s=0}^{t} (2 + d_s) \right\} \Big/ 2^t + \tfrac{1}{8} \left\{ \prod_{s=0}^{t} (2 + d_s) \right\} \left\{ 2^t \cdot 2^{t+1} \prod_{s=0}^{t+1} (2 + d_s) \right\}^{-1}.$$

Now, notice that $d_{t+1}^2 + 2d_{t+1} = \Theta_t = [2^{t+2} \prod_{s=0}^{t} (2 + d_s)]^{-1}$. So,

$$F_{R_t} \cdot F_{K_{t+1}} = \tfrac{1}{2} \left\{ \prod_{s=0}^{t} (2 + d_s) \right\} \{2^{t+1}\}^{-1} [2 + \{\Theta_t/(2 + d_{t+1})\}]$$

$$= \tfrac{1}{2} \left\{ \prod_{s=0}^{t+1} (2 + d_s) \right\} \{2^{t+1}\}^{-1} = F_{R_{t+1}}.$$

Hence, by Proposition 3.1, $\langle K, R, L, Y, C \rangle$ is competitive. To check (ii), note that the manner in which M is defined ensures that $\sum_{t=0}^{\infty} R_t = M = \mathbf{S}$.

Finally, note that the capital-value transversality condition, is violated, since $p_{t+1}K_{t+1} = 1/F_{R_t} = 2R_t^{1/2}L_t^{-1/4} = 2^{t+1}[\prod_{s=0}^{t} (2 + d_s)]^{-1} = [\prod_{s=0}^{t} (1 + \tfrac{1}{2}d_s)]^{-1} \geqslant 1/\tilde{\Theta}$ (for some $0 < \tilde{\Theta} < \infty$), for $t \geqslant 0$, as $\sum_{t=0}^{\infty} \tfrac{1}{2}d_t \leqslant 1$. To check this last fact, observe that $d_{t+1}^2 + 2d_{t+1} = \Theta_t \leqslant [2^{t+2}]^{-1}$ for $t \geqslant 0$, so that $d_{t+1} \leqslant [2^{t+2}]^{-1}$, and the series $\sum_{t+0}^{\infty} d_t$ is therefore convergent, with a sum $\leqslant 2$.

6. On the Possibility of Competitive Overaccumulation of Capital

If there is competitive overaccumulation of capital, in the present framework, it is signaled by the capital-value transversality condition being violated. Example 5.2 should give the idea to the reader that such overaccumulation is indeed a rare phenomenon, as the circumstances under which it is shown to exist are hardly likely to occur. It turns out, interestingly enough, that with some additional structure on the model, competitive overaccumulation of capital cannot occur at all.

We assume, in this section, that

(A4) $G(K, 0, L) = 0$ *for all* $(K, L) \geqslant 0$,

(A5) *For* $(K, L) \geqslant (\gamma, \gamma)$, *where* $\gamma > 0$, $G_R \to \infty$ *as* $R \to 0$; *for* $(R, L) \geqslant (\beta, \beta)$, *where* $\beta > 0$, $G_K \to \infty$ *as* $K \to 0$.

(A4) states that the resource is indispensable in production; (A5) is similar to the "Inada conditions" on the production function, so well known in the literature on growth theory.

In addition, we shall assume

(A6) *There exist positive numbers* m, $\bar{m}$, *such that* $m \leqslant \mathbf{L}_t \leqslant \bar{m}$ *for all* $t \geqslant 0$.

The lower bound, m, in (A6) hardly needs justification. The upper bound $\bar{m}$ rules out unbounded labor force, which seems to be an appropriate restriction in a model with exhaustible resources. (See Solow [8, p. 36] for a discussion of this assumption.)

THEOREM 6.1. *Under* (A1)–(A6), *an interior program* $\langle K, R, L, Y, C\rangle$ *is efficient if and only if*

(a) *it is competitive, and*

(b) *it satisfies the resource exhaustion condition.*

Proof. (Necessity) This is clear from the proof of Theorem 4.1.

(*Sufficiency*) $\langle K, R, L, Y, C\rangle$ is competitive at the price sequence $\langle p, q, w\rangle$ defined by (3.4). I shall show that, at this price sequence, $\lim_{t\to\infty} \inf p_t K_t = 0$. I shall break up the proof into three cases.

[8] The idea of the example is so obvious, (namely, take any efficient program with $\mathbf{S} = S^1$, and suppose, instead, that $\mathbf{S} = S^2 > S^1$) that the reader may wonder why I have bothered to specify it, in some detail. A technical reason is that the example should be constructed for *interior* competitive programs, since all the results are for interior programs. This, immediately, makes the construction nontrivial. An expository reason is that, in view of Corollary 6.1, the example assumes so much importance, that its details, even if obvious, become worth presenting.

Case 1 ($\delta = 0$). I claim that $\lim_{t\to\infty} \inf p_{t+1}K_t = 0$. For suppose this were not true, then there exists $\hat{\alpha} > 0$, such that $p_{t+1}K_t \geqslant \hat{\alpha}$ for $t \geqslant 0$. Using (A3), some routine calculations yield $G_{K_t} \leqslant R_t/\alpha\hat{\alpha}$. So that $\sum_{t=1}^{\infty} G_{K_t}$ is bounded above. It follows that $G_{R_t} = G_{R_0} \prod_{s=1}^{t} (1 + G_{K_s})$ is also bounded above. But, $K_t \geqslant \hat{\alpha} G_{R_t} \geqslant \hat{\alpha} G_{R_0}$, and $L_t \geqslant m$, so, by (A5), $G_{R_t} \to \infty$, since $R_t \to 0$. This contradiction establishes the claim.

Next, I claim that $\lim_{t\to\infty} \inf p_t K_t = 0$. Suppose this were not true, then there exists $\hat{\Theta} > 0$, such that $p_t K_t \geqslant \hat{\Theta}$ for $t \geqslant 0$. Since $K_t \geqslant \hat{\Theta} G_{R_{t-1}} \geqslant \hat{\Theta} G_{R_0}$, and $L_t \geqslant m$, so by (A5), $G_{R_t} \to \infty$ since $R_t \to 0$. So, $K_t \to$ R. Now, by (A4), there exists $\bar{K}$, such that for $K_t \geqslant \bar{K}$, $G(K_t, R_t, \bar{m})/K_t < 1$, so that for $K_t \geqslant \bar{K}$, $G_{K_t} < G(K_t, R_t, L_t)/K_t \leqslant G(K_t, R_t, \bar{m})/K_t < 1$, and $F_{K_t} \leqslant 2$. Since $K_t \to \infty$, so for $t \geqslant t'$, $F_{K_t} \leqslant 2$. Hence, for $t \geqslant t'$, $0 < \hat{\Theta} \leqslant p_t K_t \leqslant 2 p_{t+1} K_t$, which contradicts the fact that $\lim_{t\to\infty} \inf p_{t+1} K_t = 0$, and establishes the claim.

Case 2 ($0 < \delta < 1$). I shall show, in this case, that (a′) $\lim_{t\to\infty} \inf K_t = 0$, and (b′) p_t is bounded above. To show (a′), suppose it is violated. Then there is $\hat{\epsilon} > 0$ such that $K_t \geqslant \hat{\epsilon}$ for $t \geqslant 0$. Observe, now, that K_t is bounded above. For, there exists $\bar{\bar{K}}$ such that for $K_t \geqslant \bar{\bar{K}}$, $G(K_t, R_t, \bar{m})/K_t \leqslant \delta$ (by (A4)), and so for $K_t \geqslant \bar{\bar{K}}$, $K_{t+1} \leqslant F(K_t, R_t, L_t) \leqslant G(K_t, R_t, L_t) + (1 - \delta) K_t \leqslant K_t$. Also, for $K_t < \bar{\bar{K}}$, $K_{t+1} \leqslant G(\bar{\bar{K}}, \mathrm{S}, \bar{m}) + K_t = K_t + A$, where $A < \infty$. Hence $\langle K_t \rangle$ is a bounded sequence. Using this, and the fact that $R_t \to 0$ as $t \to \infty$, we can find t'', such that for $t \geqslant t''$, $G(K_t, R_t, \bar{m}) \leqslant \frac{1}{2}\delta\hat{\epsilon}$ [by (A4)]. So $K_{t+1} \leqslant F(K_t, R_t, \bar{m}) \leqslant (1 - \delta) K_t + \frac{1}{2}\delta\hat{\epsilon} \leqslant (1 - \delta) K_t + \frac{1}{2}\delta K_t = (1 - (\delta/2)) K_t$ for $t \geqslant t''$. So, $K_t \to 0$ as $t \to \infty$, a contradiction, which establishes (a′).

To show (b′), use the competitive condition, for each period, for $(K, R, L) = (\epsilon, 1, 1)$, where ϵ is small enough so that $G_K(\epsilon, 1, 1) \geqslant 1$ (by (A5), this can be done). It follows that $G(\epsilon, 1, 1)/\epsilon \geqslant G_K(\epsilon, 1, 1) \geqslant 1$, i.e., $G(\epsilon, 1, 1) \geqslant \epsilon$. Using this in (2.6) together with the fact that G is homogeneous of degree one, we have $\epsilon[1 + (1 - \delta)] p_{t+1} \leqslant G(\epsilon, 1, 1) + (1 - \delta)\epsilon \leqslant p_t \epsilon + 1 + w_t$. So $p_{t+1} \leqslant p_t/(1 + (1 - \delta)) + (1/\epsilon)/(1 + (1 - \delta)) + (w_t/\epsilon)/(1 + (1 - \delta))$. Using this recursively, and noting that $\sum_{t=0}^{\infty} w_t$ is bounded above, p_t must be bounded above, proving (b′). From (a′) and (b′), we conclude that $\lim_{t\to\infty} \inf \mathrm{p}_t K_t = 0$.

Case 3 ($\delta = 1$). Here $p_t K_t = K_t/F_{R_{t-1}} = K_t/G_{R_{t-1}}$. Use (A3) to write $p_t K_t \leqslant (K_t R_{t-1})/\alpha G(K_{t-1}, R_{t-1}, L_{t-1}) \leqslant R_{t-1}/\alpha$, so $\lim_{t\to\infty} \inf p_t K_t = 0$.

Thus, in all three cases, $\lim_{t\to\infty} \inf p_t K_t = 0$. We know from Section 4 that $\lim_{t\to\infty} p_t K_t$ exists, so $\lim_{t\to\infty} p_t K_t = 0$. Now, the theorem follows from Theorem 4.1.

From the proof of Theorem 6.1, the following interesting corollary is immediately obtained.

COROLLARY 6.1. *Under* (A1)–(A6), *an interior program* $\langle K, R, L, Y, C\rangle$, *which is competitive, satisfies the capital value transversality condition.*

Remark 6.1. Corollary 6.1, together with Example 5.1, demonstrates that only overaccumulation of the exhaustible resource, and not over-accumulation of the capital-cum-consumption good is consistent with competitive pricing.

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